DSC 140B
Representation Learning

Lecture 07 | Part 1

Why are eigenvectors useful?

OK, but why are eigenvectors[4] useful?

1. Eigenvectors are nice “building blocks” (basis vectors).

2. Eigenvectors are **maximizers** (or minimizers).

3. Eigenvectors are **equilibria**.

[4] of symmetric matrices

Vector Decomposition

- We can always “decompose” a vector $\vec{x}$ in terms of the basis vectors.

- With respect to the standard basis:

$$\vec{x} = a_1 \hat{e}^{(1)} + a_2 \hat{e}^{(2)} + \ldots + a_d \hat{e}^{(d)}$$

Eigendecomposition

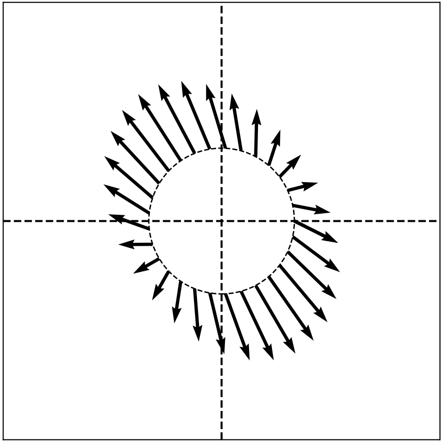
- If A is a symmetric matrix, we can pick d of its eigenvectors $\hat{u}^{(1)}, \ldots, \hat{u}^{(d)}$ to form an orthonormal basis.

- Any vector $\vec{x}$ can be written in terms of this basis.

- This is called its **eigendecomposition**:

$$\vec{x} = b_1 \hat{u}^{(1)} + b_2 \hat{u}^{(2)} + \ldots + b_d \hat{u}^{(d)}$$

Eigendecomposition



Why?

- Compare working in the standard basis decomposition:

$$
\begin{aligned}
A\vec{x} &= A(a_1 \hat{e}^{(1)} + a_2 \hat{e}^{(2)} + \ldots + a_d \hat{e}^{(d)}) \\
&= a_1 A\hat{e}^{(1)} + a_2 A\hat{e}^{(2)} + \ldots + a_d A\hat{e}^{(d)}
\end{aligned}
$$

- To working with the eigendecomposition:

$$
\begin{aligned}
A\vec{x} &= A(b_1 \hat{u}^{(1)} + b_2 \hat{u}^{(2)} + \ldots + b_d \hat{u}^{(d)}) \\
&= b_1 A\hat{u}^{(1)} + b_2 A\hat{u}^{(2)} + \ldots + b_d A\hat{u}^{(d)}) \\
&= \lambda_1 b_1 \hat{u}^{(1)} + \lambda_2 b_2 \hat{u}^{(2)} + \ldots + \lambda_d b_d \hat{u}^{(d)}
\end{aligned}
$$

Main Idea

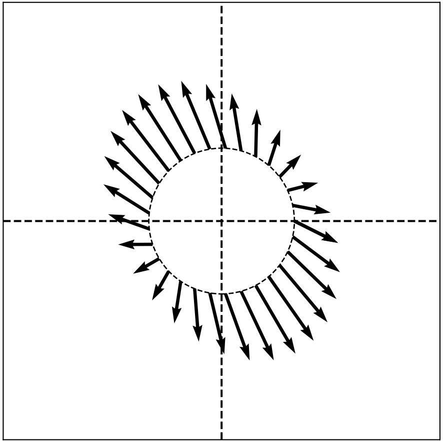
If A is a symmetrix matrix, an eigenbasis formed from its eigenvectors is an especially natural basis.

Eigenvectors as Optimizers

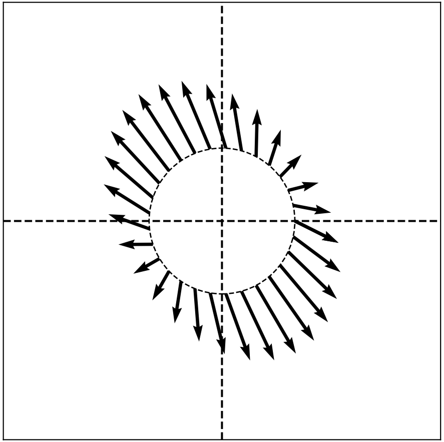
- Eigenvectors are the solutions to certain common optimization problems involving matrices / linear transformations.

Exercise

Draw a unit vector $\vec{x}$ such that $\|A\vec{x}\|$ is largest.



Observation #1



- $\vec{f}(\vec{x})$ is longest along the "main" axis of symmetry.
  - In the direction of the eigenvector with largest eigenvalue.

Main Idea

To maximize $\|A\vec{x}\| = \|\vec{f}(\vec{x})\|$ over unit vectors, pick $\vec{x}$ to be an eigenvector of $\vec{f}$ with the largest eigenvalue (in abs. value).

Main Idea

To **minimize** $\|\vec{f}(\vec{x})\|$ over unit vectors, pick $\vec{x}$ to be an eigenvector of $\vec{f}$ with the smallest eigenvalue (in abs. value).

Proof

Show that the maximizer of $\|A\vec{x}\|$ s.t., $\|\vec{x}\| = 1$ is the top eigenvector of A.

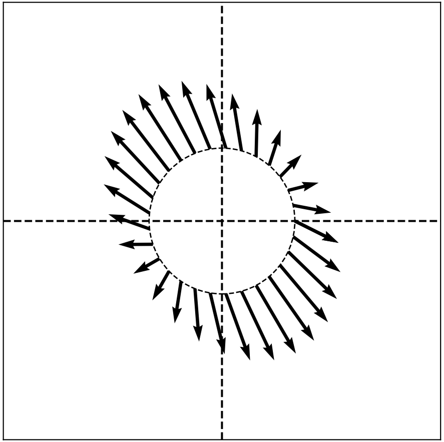
Corollary

To maximize $\vec{x} \cdot A\vec{x}$ over unit vectors, pick $\vec{x}$ to be top eigenvector of A.

Example

- Maximize $4x_1^2 + 2x_2^2 + 3x_1x_2$ subject to $x_1^2 + x_2^2 = 1$

Observation #2



- $\vec{f}(\vec{x})$ rotates $\vec{x}$ towards the "top" eigenvector $\vec{v}$.
- $\vec{v}$ is an equilibrium.

The Power Method

- Method for computing the top eigenvector/value of A.
- Initialize $\vec{x}^{(0)}$ randomly
- Repeat until convergence:
  - Set $\vec{x}^{(i+1)} = A\vec{x}^{(i)} / \|A\vec{x}^{(i)}\|$

DSC 140B
Representation Learning

Lecture 07 | Part 2

Change of Basis Matrics

Changing Basis

- Suppose $\vec{x} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = a_1 \hat{e}^{(1)} + a_2 \hat{e}^{(2)}$.

- $\hat{u}^{(1)}$ and $\hat{u}^{(2)}$ form a new, **orthonormal** basis $\mathcal{U}$.

- What is $[\vec{x}]_{\mathcal{U}}$?

- That is, what are b_1 and b_2 in $\vec{x} = b_1 \hat{u}^{(1)} + b_2 \hat{u}^{(2)}$.

Exercise

Find the coordinates of $\vec{x}$ in the new basis:

$$\hat{u}^{(1)} = (\sqrt{3}/2, 1/2)^T$$
$$\hat{u}^{(2)} = (-1/2, \sqrt{3}/2)^T$$
$$\vec{x} = (1/2, 1)^T$$

Change of Basis

- Suppose $\hat{u}^{(1)}$ and $\hat{u}^{(2)}$ are our new, **orthonormal** basis vectors.
- We know $\vec{x} = x_1 \hat{e}^{(1)} + x_2 \hat{e}^{(2)}$
- We want to write $\vec{x} = b_1 \hat{u}^{(1)} + b_2 \hat{u}^{(2)}$
- Solution

$$b_1 = \vec{x} \cdot \hat{u}^{(1)} \qquad b_2 = \vec{x} \cdot \hat{u}^{(2)}$$

Change of Basis Matrix

- Changing basis is a linear transformation

$$\vec{f}(\vec{x}) = (\vec{x} \cdot \hat{u}^{(1)})\hat{u}^{(1)} + (\vec{x} \cdot \hat{u}^{(2)})\hat{u}^{(2)} = \begin{pmatrix} \vec{x} \cdot \hat{u}^{(1)} \\ \vec{x} \cdot \hat{u}^{(2)} \end{pmatrix}_{\mathcal{U}}$$

- We can represent it with a matrix

$$\begin{pmatrix} \uparrow & \uparrow \\ f(\hat{e}^{(1)}) & f(\hat{e}^{(2)}) \\ \downarrow & \downarrow \end{pmatrix}$$

Example

$$\hat{u}^{(1)} = (\sqrt{3}/2, 1/2)^T$$

$$\hat{u}^{(2)} = (-1/2, \sqrt{3}/2)^T$$

$$f(\hat{e}^{(1)}) =$$

$$f(\hat{e}^{(2)}) =$$

$$A =$$

Observation

- The new basis vectors become the **rows** of the matrix.

Example

- Multiplying by this matrix gives the coordinate vector w.r.t. the new basis.

$$\hat{u}^{(1)} = (\sqrt{3}/2, 1/2)^T$$

$$\hat{u}^{(2)} = (-1/2, \sqrt{3}/2)^T$$

$$A = \begin{pmatrix} \sqrt{3}/2 & 1/2 \\ -1/2 & \sqrt{3}/2 \end{pmatrix}$$

$$\vec{x} = (1/2, 1)^T$$

Change of Basis Matrix

- Let $\hat{u}^{(1)}, \ldots, \hat{u}^{(d)}$ form an orthonormal basis $\mathcal{U}$.

- The matrix U whose **rows** are the new basis vectors is the **change of basis** matrix from the standard basis to $\mathcal{U}$:

$$U = \begin{pmatrix} \leftarrow & \hat{u}^{(1)} & \rightarrow \\ \leftarrow & \hat{u}^{(2)} & \rightarrow \\ & \vdots & \\ \leftarrow & \hat{u}^{(d)} & \rightarrow \end{pmatrix}$$

Change of Basis Matrix

- If U is the change of basis matrix, $[\vec{x}]_{\mathcal{U}} = U\vec{x}$
- To go *back* to the standard basis, use U^T:

$$\vec{x} = U^T[\vec{x}]_{\mathcal{U}}$$

Exercise

Let U be the change of basis matrix for $\mathcal{U}$.
What is $U^T U$?

Hint: What is $U^T(U\vec{x})$?

DSC 140B

Representation Learning

Lecture 07 | Part 3

Diagonalization

Matrices of a Transformation

- Let $\vec{f} : \mathbb{R}^d \to \mathbb{R}^d$ be a linear transformation

- The matrix representing $\vec{f}$ wrt the **standard basis** is:

$$A = \begin{pmatrix} \uparrow & \uparrow & \uparrow & \uparrow \\ \vec{f}(\hat{e}^{(1)}) & \vec{f}(\hat{e}^{(2)}) & \cdots & \vec{f}(\hat{e}^{(d)}) \\ \downarrow & \downarrow & \downarrow & \downarrow \end{pmatrix}$$

Matrices of a Transformation

- If we use a different basis $\mathcal{U} = \{\hat{u}^{(1)}, \ldots, \hat{u}^{(d)}\}$, the matrix representing $\vec{f}$ is:

$$A_{\mathcal{U}} = \begin{pmatrix} \uparrow & \uparrow & \uparrow & \uparrow \\ [\vec{f}(\hat{u}^{(1)})]_{\mathcal{U}} & [\vec{f}(\hat{u}^{(2)})]_{\mathcal{U}} & \cdots & [\vec{f}(\hat{u}^{(d)})]_{\mathcal{U}} \\ \downarrow & \downarrow & \downarrow & \downarrow \end{pmatrix}$$

- If $\vec{y} = A\vec{x}$, then $[\vec{y}]_{\mathcal{U}} = A_{\mathcal{U}}[\vec{x}]_{\mathcal{U}}$

Diagonal Matrices

- Diagonal matrices are very nice / easy to work with.
- Suppose A is a matrix. Is there a basis $\mathcal{U}$ where $A_{\mathcal{U}}$ is diagonal?
- Yes! *If* A is symmetric.

The Spectral Theorem[1]

- **Theorem**: Let A be an $n \times n$ *symmetric* matrix. Then there exist n eigenvectors of A which are all mutually orthogonal.

[1] for symmetric matrices

Eigendecomposition

- If A is a symmetric matrix, we can pick d of its eigenvectors $\hat{u}^{(1)}, \ldots, \hat{u}^{(d)}$ to form an orthonormal basis.

- Any vector $\vec{x}$ can be written in terms of this **eigenbasis**.

- This is called its **eigendecomposition**:

$$\vec{x} = b_1 \hat{u}^{(1)} + b_2 \hat{u}^{(2)} + \ldots + b_d \hat{u}^{(d)}$$

Matrix in the Eigenbasis

- **Claim**: the matrix of a linear transformation $\vec{f}$, written in a basis of its eigenvectors, is a **diagonal** matrix.

- The entries along the diagonal will be the **eigenvalues**.

Why?

$$A_{\mathcal{U}} = \begin{pmatrix} \uparrow & \uparrow & \uparrow & \uparrow \\ [\vec{f}(\hat{u}^{(1)})]_{\mathcal{U}} & [\vec{f}(\hat{u}^{(2)})]_{\mathcal{U}} & \cdots & [\vec{f}(\hat{u}^{(d)})]_{\mathcal{U}} \\ \downarrow & \downarrow & \downarrow & \downarrow \end{pmatrix}$$

- $\vec{f}(\hat{u}^{(1)}) = \lambda_1 \hat{u}^{(1)}$, so $[\vec{f}(\hat{u}^{(1)})]_{\mathcal{U}} = (\lambda_1, 0, \ldots, 0)^T$.
- $\vec{f}(\hat{u}^{(2)}) = \lambda_2 \hat{u}^{(2)}$, so $[\vec{f}(\hat{u}^{(2)})]_{\mathcal{U}} = (0, \lambda_2, \ldots, 0)^T$.
- ...

Matrix Multiplication

- We have seen that matrix multiplication evaluates a linear transformation.
- In the standard basis:

$$\vec{f}(\vec{x}) = A\vec{x}$$

- In another basis:

$$[\vec{f}(\vec{x})]_{\mathcal{U}} = A_{\mathcal{U}}[\vec{x}]_{\mathcal{U}}$$

Diagonalization

- Another way to compute $\vec{f}(x)$, starting with $\vec{x}$ in the standard basis:
  1. Change basis to the eigenbasis with U.
  2. Apply $\vec{f}$ in the eigenbasis with the diagonal $A_{\mathcal{U}}$.
  3. Go *back* to the standard basis with U^T.
- That is, $A\vec{x} = U^T A_{\mathcal{U}} U \vec{x}$. It follows that $A = U^T A_{\mathcal{U}} U$.

Spectral Theorem (Again)

- **Theorem**: Let A be an $n \times n$ *symmetric* matrix. Then there exists an orthogonal matrix U and a diagonal matrix Λ such that $A = U^T \Lambda U$.
- The *rows* of U are the eigenvectors of A, and the entries of Λ are its eigenvalues.
- U is said to **diagonalize** A.