DSC 140B
Representation Learning

Lecture 07 | Part 1

Why are eigenvectors useful?

OK, but why are eigenvectors[4] useful?

1. Eigenvectors are nice “building blocks” (basis vectors).

2. Eigenvectors are **maximizers** (or minimizers).

3. Eigenvectors are **equilibria**.

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[4] of symmetric matrices

Vector Decomposition

- We can always "decompose" a vector $\vec{x}$ in terms of the basis vectors.
- With respect to the standard basis:

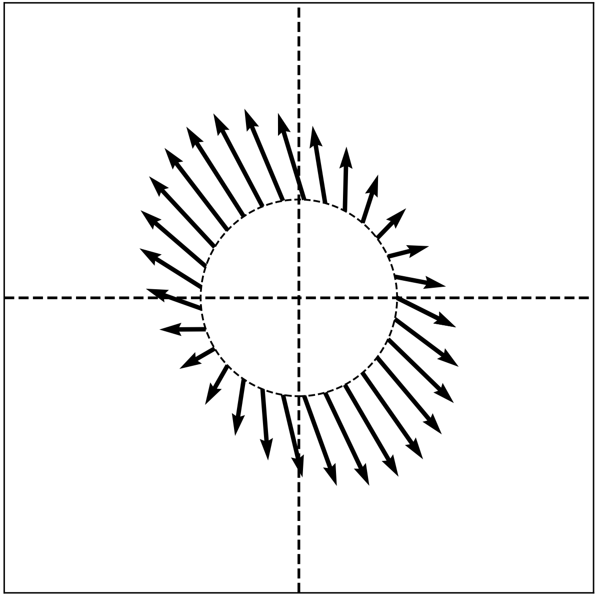
$$\vec{x} = a_1 \hat{e}^{(1)} + a_2 \hat{e}^{(2)} + \ldots + a_d \hat{e}^{(d)}$$

Eigendecomposition

- If A is a symmetric matrix, we can pick d of its eigenvectors $\hat{u}^{(1)}, \ldots, \hat{u}^{(d)}$ to form an orthonormal basis.
- Any vector $\vec{x}$ can be written in terms of this basis.
- This is called its **eigendecomposition**:

$$\vec{x} = b_1 \hat{u}^{(1)} + b_2 \hat{u}^{(2)} + \ldots + b_d \hat{u}^{(d)}$$

Eigendecomposition



Why?

- Compare working in the standard basis decomposition:

$$
\begin{aligned}
A\vec{x} &= A(a_1\hat{e}^{(1)} + a_2\hat{e}^{(2)} + \ldots + a_d\hat{e}^{(d)}) \\
&= a_1A\hat{e}^{(1)} + a_2A\hat{e}^{(2)} + \ldots + a_dA\hat{e}^{(d)}
\end{aligned}
$$

- To working with the eigendecomposition:

$$
\begin{aligned}
A\vec{x} &= A(b_1\hat{u}^{(1)} + b_2\hat{u}^{(2)} + \ldots + b_d\hat{u}^{(d)}) \\
&= b_1A\hat{u}^{(1)} + b_2A\hat{u}^{(2)} + \ldots + b_dA\hat{u}^{(d)}) \\
&= \lambda_1b_1\hat{u}^{(1)} + \lambda_2b_2\hat{u}^{(2)} + \ldots + \lambda_db_d\hat{u}^{(d)}
\end{aligned}
$$

Main Idea

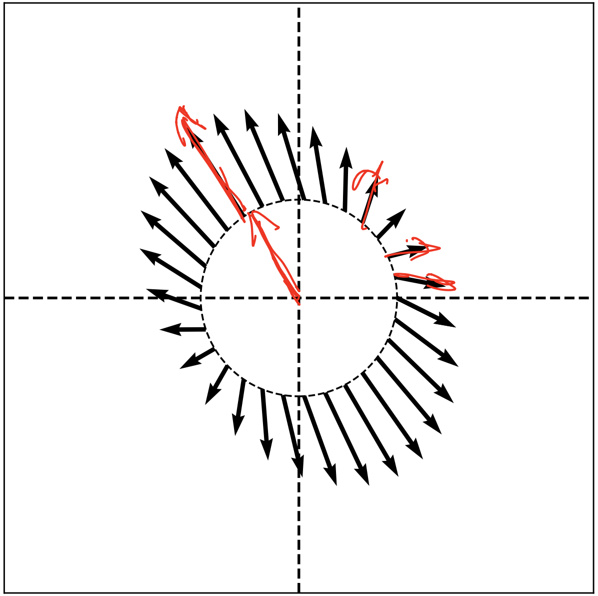
If A is a symmetrix matrix, an eigenbasis formed from its eigenvectors is an especially natural basis.

Eigenvectors as Optimizers

- Eigenvectors are the solutions to certain common optimization problems involving matrices / linear transformations.

Exercise

Draw a unit vector $\vec{x}$ such that $\|A\vec{x}\|$ is largest.

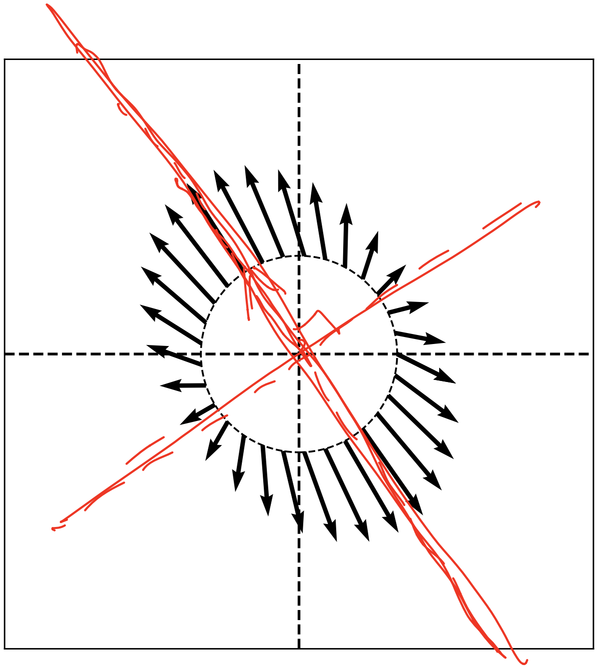


$\|A\vec{x}\|$

$\vec{x}$:

A

Observation #1



- $\vec{f}(\vec{x})$ is longest along the "main" axis of symmetry. = eigen vector
  - In the direction of the eigenvector with largest eigenvalue.

Main Idea

To maximize $\|A\vec{x}\| = \|\vec{f}(\vec{x})\|$ over unit vectors, pick $\vec{x}$ to be an eigenvector of $\vec{f}$ with the largest eigenvalue (in abs. value).

Main Idea

To **minimize** $\|\vec{f}(\vec{x})\|$ over unit vectors, pick $\vec{x}$ to be an eigenvector of $\vec{f}$ with the smallest eigenvalue (in abs. value).

Proof

eigen decomposition

assume A: $\vec{u}^{(1)}$ $\vec{u}^{(2)}$

$\|b_1\vec{u}^{(1)}+b_2\vec{u}^{(2)}\|^2=1 \Rightarrow b_1^2+b_2^2=1$

A symmetric

Show that the maximizer of $\|A\vec{x}\|$ s.t., $\boxed{\|\vec{x}\| = 1}$ is the top eigenvector of A.

$$A\vec{x} = A(b_1\vec{u}^{(1)}+b_2\vec{u}^{(2)})$$
$$= b_1A\vec{u}^{(1)}+b_2A\vec{u}^{(2)}$$
$$= b_1\lambda_1\vec{u}^{(1)}+b_2\lambda_2\vec{u}^{(2)}$$

$$\|A\vec{x}\|^2 = (b_1\lambda_1\vec{u}^{(1)}+b_2\lambda_2\vec{u}^{(2)})\cdot(b_1\lambda_1\vec{u}^{(1)}+b_2\lambda_2\vec{u}^{(2)})$$
$$\Rightarrow = b_1^2\lambda_1^2\underbrace{\vec{u}^{(1)T}\vec{u}^{(1)}}_{1}+2b_1b_2\lambda_1\lambda_2\underbrace{\vec{u}^{(1)T}\vec{u}^{(2)}}_{0}+b_2^2\lambda_2^2\underbrace{\vec{u}^{(2)T}\vec{u}^{(2)}}_{1}$$

$$\max\ b_1^2\lambda_1^2+b_2^2\lambda_2^2$$

assume $\lambda_1^2 \geq \lambda_2^2$

$\Rightarrow b_1=1 \quad b_2=0$

Corollary

To maximize $\vec{x} \cdot A\vec{x}$ over unit vectors, pick $\vec{x}$ to be top eigenvector of A.

$$\| b_1 \vec{u}^{(1)} + b_2 \vec{u}^{(2)} \|^2 = 1$$

$$(b_1 \vec{u}^{(1)} + b_2 \vec{u}^{(2)})^T (b_1 \vec{u}^{(1)} + b_2 \vec{u}^{(2)})$$

$$= b_1^2 \underbrace{\vec{u}^{(1)T} \vec{u}^{(1)}}_{1} + 2 b_1 b_2 \underbrace{\vec{u}^{(1)T} \vec{u}^{(2)}}_{0} + b_2^2 \underbrace{\vec{u}^{(2)T} \vec{u}^{(2)}}_{1}$$

$$= b_1^2 + b_2^2$$

Example

$\vec{x} = (x_1, x_2)$

$\|\vec{x}\| = 1$

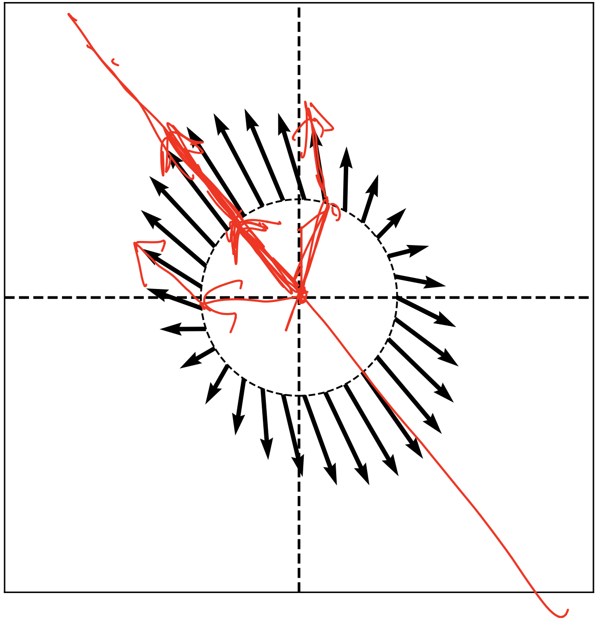
- Maximize $4x_1^2 + 2x_2^2 + 3x_1x_2$ subject to $x_1^2 + x_2^2 = 1$

$$\underbrace{(x_1 \;,\; x_2)}_{\vec{x}^T} \underbrace{\begin{pmatrix} 4 & 1.5 \\ 1.5 & 2 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}}_{\vec{x}}$$

$\max \vec{x} \cdot A \cdot \vec{x}$

$\|\vec{x}\| = 1$

Observation #2



- $\vec{f}(\vec{x})$ rotates $\vec{x}$ towards the "top" eigenvector $\vec{v}$.
- $\vec{v}$ is an equilibrium.

The Power Method

- Method for computing the top eigenvector/value of A.
- Initialize $\vec{x}^{(0)}$ randomly
- Repeat until convergence:
  - Set $\vec{x}^{(i+1)} = A\vec{x}^{(i)} / \|A\vec{x}^{(i)}\|$

$\vec{f}(\vec{x})$

$A\vec{x}$

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Representation Learning

Lecture 07 | Part 2

Change of Basis Matrics

Changing Basis

- Suppose $\vec{x} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = a_1 \hat{e}^{(1)} + a_2 \hat{e}^{(2)}$.

- $\hat{u}^{(1)}$ and $\hat{u}^{(2)}$ form a new, **orthonormal** basis $\mathcal{U}$.

- What is $[\vec{x}]_{\mathcal{U}}$?

- That is, what are b_1 and b_2 in $\vec{x} = b_1 \hat{u}^{(1)} + b_2 \hat{u}^{(2)}$.

$$[\vec{x}]_u = \begin{pmatrix} \vec{x} \cdot \hat{u}^{(1)} \\ \vec{x} \cdot \hat{u}^{(2)} \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{3}}{4} + \frac{1}{2} \\ -\frac{1}{4} + \frac{\sqrt{3}}{2} \end{pmatrix}$$

Exercise

Find the coordinates of $\vec{x}$ in the new basis:

$$\hat{u}^{(1)} = (\sqrt{3}/2, 1/2)^T$$
$$\hat{u}^{(2)} = (-1/2, \sqrt{3}/2)^T$$
$$\vec{x} = (1/2, 1)^T$$

Change of Basis

- Suppose $\hat{u}^{(1)}$ and $\hat{u}^{(2)}$ are our new, **orthonormal** basis vectors.
- We know $\vec{x} = x_1 \hat{e}^{(1)} + x_2 \hat{e}^{(2)}$
- We want to write $\vec{x} = b_1 \hat{u}^{(1)} + b_2 \hat{u}^{(2)}$
- Solution

$$b_1 = \vec{x} \cdot \hat{u}^{(1)} \qquad b_2 = \vec{x} \cdot \hat{u}^{(2)}$$

Change of Basis Matrix

- Changing basis is a linear transformation

$$\vec{f}(\vec{x}) = (\underbrace{\vec{x} \cdot \hat{u}^{(1)}}_{b_1})\hat{u}^{(1)} + (\underbrace{\vec{x} \cdot \hat{u}^{(2)}}_{b_2})\hat{u}^{(2)} = \begin{pmatrix} \vec{x} \cdot \hat{u}^{(1)} \\ \vec{x} \cdot \hat{u}^{(2)} \end{pmatrix}_{\mathcal{U}}$$

- We can represent it with a matrix

$$\begin{pmatrix} \uparrow & \uparrow \\ f(\hat{e}^{(1)}) & f(\hat{e}^{(2)}) \\ \downarrow & \downarrow \end{pmatrix}$$

Example

$\hat{u}^{(1)} = (\sqrt{3}/2, 1/2)^T$

$\hat{u}^{(2)} = (-1/2, \sqrt{3}/2)^T$

$f(\hat{e}^{(1)}) =$

$f(\hat{e}^{(2)}) =$

$A = \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$

$f(\hat{e}^{(1)})$ $f(\hat{e}^{(2)})$

$$f(\hat{e}^{(1)}) = \begin{pmatrix} \hat{e}^{(1)} \cdot \hat{u}^{(1)} \\ \hat{e}^{(2)} \cdot \hat{u}^{(2)} \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{3}}{2} \\ -\frac{1}{2} \end{pmatrix}$$

$$f(\hat{e}^{(2)}) = \begin{pmatrix} \hat{e}^{(2)} \cdot \hat{u}^{(1)} \\ \hat{e}^{(2)} \cdot \hat{u}^{(2)} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix}$$

Observation

- The new basis vectors become the **rows** of the matrix.

Example

$$[\vec{x}]_u = \begin{pmatrix} \vec{x}\cdot\vec{u}^{(1)} \\ \vec{x}\cdot\vec{u}^{(2)} \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{3}}{4}+\frac{1}{2} \\ -\frac{1}{4}+\frac{\sqrt{3}}{2} \end{pmatrix}$$

- Multiplying by this matrix gives the coordinate vector w.r.t. the new basis.

$$\hat{u}^{(1)} = (\sqrt{3}/2, 1/2)^T$$

$$\hat{u}^{(2)} = (-1/2, \sqrt{3}/2)^T$$

$$A = \begin{pmatrix} \sqrt{3}/2 & 1/2 \\ -1/2 & \sqrt{3}/2 \end{pmatrix}$$

$$\vec{x} = (1/2, 1)^T$$

$$A\cdot\vec{x} = \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{-1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}\begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{3}}{4}+\frac{1}{2} \\ -\frac{1}{4}+\frac{\sqrt{3}}{2} \end{pmatrix}$$

Change of Basis Matrix

- Let $\hat{u}^{(1)}, \ldots, \hat{u}^{(d)}$ form an orthonormal basis $\mathcal{U}$.
- The matrix U whose **rows** are the new basis vectors is the **change of basis** matrix from the standard basis to $\mathcal{U}$:

$$U = \begin{pmatrix} \leftarrow & \hat{u}^{(1)} & \rightarrow \\ \leftarrow & \hat{u}^{(2)} & \rightarrow \\ & \vdots & \\ \leftarrow & \hat{u}^{(d)} & \rightarrow \end{pmatrix}$$

Change of Basis Matrix

- If U is the change of basis matrix, $[\vec{x}]_{\mathcal{U}} = U\vec{x}$
- To go *back* to the standard basis, use U^T:

$$\vec{x} = U^T[\vec{x}]_{\mathcal{U}}$$

$$U^T[\vec{x}]_{\mathcal{U}} = U^T(U \cdot \vec{x}) = (\underbrace{U^T U}_{\Downarrow\, I}) \cdot \vec{x} = \vec{x}$$

I : identity matrix

Exercise

Let U be the change of basis matrix for $\mathcal{U}$.
What is $U^T U$?

Hint: What is $U^T(U\vec{x})$?

DSC 140B
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Lecture 07 | Part 3

Diagonalization

Matrices of a Transformation

- Let $\vec{f} : \mathbb{R}^d \to \mathbb{R}^d$ be a linear transformation
- The matrix representing $\vec{f}$ wrt the **standard basis** is:

$$A = \begin{pmatrix} \uparrow & \uparrow & \uparrow & \uparrow \\ \vec{f}(\hat{e}^{(1)}) & \vec{f}(\hat{e}^{(2)}) & \cdots & \vec{f}(\hat{e}^{(d)}) \\ \downarrow & \downarrow & \downarrow & \downarrow \end{pmatrix}$$

Matrices of a Transformation

- If we use a different basis $\mathcal{U} = \{\hat{u}^{(1)}, \ldots, \hat{u}^{(d)}\}$, the matrix representing $\vec{f}$ is:

$$A_{\mathcal{U}} = \begin{pmatrix} \uparrow & \uparrow & \uparrow & \uparrow \\ [\vec{f}(\hat{u}^{(1)})]_{\mathcal{U}} & [\vec{f}(\hat{u}^{(2)})]_{\mathcal{U}} & \cdots & [\vec{f}(\hat{u}^{(d)})]_{\mathcal{U}} \\ \downarrow & \downarrow & \downarrow & \downarrow \end{pmatrix}$$

- If $\vec{y} = A\vec{x}$, then $[\vec{y}]_{\mathcal{U}} = A_{\mathcal{U}}[\vec{x}]_{\mathcal{U}}$

Diagonal Matrices

- Diagonal matrices are very nice / easy to work with.
- Suppose A is a matrix. Is there a basis $\mathcal{U}$ where $A_{\mathcal{U}}$ is diagonal?
- Yes! *If* A is symmetric.

The Spectral Theorem[1]

- **Theorem**: Let A be an $n \times n$ *symmetric* matrix. Then there exist n eigenvectors of A which are all mutually orthogonal.

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[1]for symmetric matrices

Eigendecomposition

- If A is a symmetric matrix, we can pick d of its eigenvectors $\hat{u}^{(1)}, \ldots, \hat{u}^{(d)}$ to form an orthonormal basis.
- Any vector $\vec{x}$ can be written in terms of this **eigenbasis**.
- This is called its **eigendecomposition**:

$$\vec{x} = b_1 \hat{u}^{(1)} + b_2 \hat{u}^{(2)} + \ldots + b_d \hat{u}^{(d)}$$

Matrix in the Eigenbasis

- **Claim**: the matrix of a linear transformation $\vec{f}$, written in a basis of its eigenvectors, is a **diagonal** matrix.

- The entries along the diagonal will be the **eigenvalues**.

Why?

$$A_{\mathcal{U}} = \begin{pmatrix} \uparrow & \uparrow & \uparrow & \uparrow \\ [\vec{f}(\hat{u}^{(1)})]_{\mathcal{U}} & [\vec{f}(\hat{u}^{(2)})]_{\mathcal{U}} & \cdots & [\vec{f}(\hat{u}^{(d)})]_{\mathcal{U}} \\ \downarrow & \downarrow & \downarrow & \downarrow \end{pmatrix}$$

- $\vec{f}(\hat{u}^{(1)}) = \lambda_1 \hat{u}^{(1)}$, so $[\vec{f}(\hat{u}^{(1)})]_{\mathcal{U}} = (\lambda_1, 0, \ldots, 0)^T$.
- $\vec{f}(\hat{u}^{(2)}) = \lambda_2 \hat{u}^{(2)}$, so $[\vec{f}(\hat{u}^{(2)})]_{\mathcal{U}} = (0, \lambda_2, \ldots, 0)^T$.
- ...

Matrix Multiplication

- We have seen that matrix multiplication evaluates a linear transformation.
- In the standard basis:

$$\vec{f}(\vec{x}) = A\vec{x}$$

- In another basis:

$$[\vec{f}(\vec{x})]_{\mathcal{U}} = A_{\mathcal{U}}[\vec{x}]_{\mathcal{U}}$$

Diagonalization

- Another way to compute $\vec{f}(x)$, starting with $\vec{x}$ in the standard basis:
  1. Change basis to the eigenbasis with U.
  2. Apply $\vec{f}$ in the eigenbasis with the diagonal $A_{\mathcal{U}}$.
  3. Go *back* to the standard basis with U^T.
- That is, $A\vec{x} = U^T A_{\mathcal{U}} U \vec{x}$. It follows that $A = U^T A_{\mathcal{U}} U$.

Spectral Theorem (Again)

- **Theorem:** Let A be an $n \times n$ *symmetric* matrix. Then there exists an orthogonal matrix U and a diagonal matrix Λ such that $A = U^T \Lambda U$.

- The *rows* of U are the eigenvectors of A, and the entries of Λ are its eigenvalues.

- U is said to **diagonalize** A.