DSC 140B
Representation Learning

Lecture 05 | Part 1

Matrix

Matrices?

- I thought this week was supposed to be about linear algebra... Where are the matrices?

Matrices?

- I thought this week was supposed to be about linear algebra... Where are the matrices?
- What is a matrix, anyways?

What is a matrix?

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

Recall: Linear Transformations

- A **transformation** $\vec{f}(\vec{x})$ is a function which takes a vector as input and returns a vector of the same dimensionality.

- A transformation $\vec{f}$ is **linear** if

$$\vec{f}(\alpha\vec{u} + \beta\vec{v}) = \alpha\vec{f}(\vec{u}) + \beta\vec{f}(\vec{v})$$

Recall: Linear Transformations

- **Key** consequence of **linearity**: to compute $\vec{f}(\vec{x})$, only need to know what $\vec{f}$ does to basis vectors.

- Example:

$$\vec{x} = 3\hat{e}^{(1)} - 4\hat{e}^{(2)} = \begin{pmatrix} 3 \\ -4 \end{pmatrix}$$

$$\vec{f}(\hat{e}^{(1)}) = -\hat{e}^{(1)} + 3\hat{e}^{(2)} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$$

$$\vec{f}(\hat{e}^{(2)}) = 2\hat{e}^{(1)} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

$$\vec{f}(\vec{x}) = \vec{f}\left(\begin{pmatrix} 3 \\ -4 \end{pmatrix}\right) = 3\vec{f}(\vec{e}^{(1)}) - 4\vec{f}(\vec{e}^{(2)}) = \begin{pmatrix} -11 \\ 9 \end{pmatrix}$$

Matrices

- **Idea**: Since $\vec{f}$ is defined by what it does to basis, place $\vec{f}(\hat{e}^{(1)})$, $\vec{f}(\hat{e}^{(2)})$, ... into a table as columns

- This is the **matrix** representing[2] $\vec{f}$

$$\vec{f}(\hat{e}^{(1)}) = -\hat{e}^{(1)} + 3\hat{e}^{(2)} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$$

$$\vec{f}(\hat{e}^{(2)}) = 2\hat{e}^{(1)} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -1 & 2 \\ 3 & 0 \end{pmatrix}$$

[2]with respect to the standard basis $\hat{e}^{(1)}, \hat{e}^{(2)}$

Exercise

Write the matrix representing $\vec{f}$ with respect to the standard basis, given:

$$\vec{f}(\hat{e}^{(1)}) = (1, 4, 7)^T$$
$$\vec{f}(\hat{e}^{(2)}) = (2, 5, 7)^T$$
$$\vec{f}(\hat{e}^{(3)}) = (3, 6, 9)^T$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 7 & 9 \end{pmatrix}$$

Exercise

Suppose $\vec{f}$ has the matrix below:

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

Let $\vec{x} = (-2, 1, 3)^T$. What is $\vec{f}(\vec{x})$?

$$\vec{f}(\vec{x}) = \vec{f}\begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix} = -2\vec{f}(\vec{e}^{(1)}) + \vec{f}(\vec{e}^{(2)}) + 3\vec{f}(\vec{e}^{(3)})$$

$$\begin{pmatrix} 9 \\ 15 \\ 21 \end{pmatrix}$$

Main Idea

A square $(n \times n)$ matrix can be interpreted as a compact representation of a linear transformation $f : \mathbb{R}^n \to \mathbb{R}^n$.

What is matrix multiplication?

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} \ \\ \ \\ \ \end{pmatrix}$$

$= (1)\times(-2) + (2)\times(1) + (3)\times(3)$

$= (4)\times(-2) + (5)\times(1) + (6)\times(3)$

$n \times k \quad k \times m \Rightarrow n \times m$

$3 \times 3 \quad 3 \times 1 \Rightarrow 3 \times 1$

$n \times n \quad n \times 1 \Rightarrow n \times 1$

A low-level definition

$$(A\vec{x})_i = \sum_{j=1}^{n} A_{ij} x_j$$

A low-level interpretation

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix} = -2 \begin{pmatrix} 1 \\ 4 \\ 7 \end{pmatrix} + 1 \begin{pmatrix} 2 \\ 5 \\ 8 \end{pmatrix} + 3 \begin{pmatrix} 3 \\ 6 \\ 9 \end{pmatrix}$$

In general...

$$\begin{pmatrix} \uparrow & \uparrow & \uparrow \\ \vec{a}^{(1)} & \vec{a}^{(2)} & \vec{a}^{(3)} \\ \downarrow & \downarrow & \downarrow \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_1 \vec{a}^{(1)} + x_2 \vec{a}^{(2)} + x_3 \vec{a}^{(3)}$$

Matrix Multiplication

$$\vec{x} = x_1 \hat{e}^{(1)} + x_2 \hat{e}^{(2)} + x_3 \hat{e}^{(3)} = (x_1, x_2, x_3)^T$$

$$\vec{f}(\vec{x}) = x_1 \vec{f}(\hat{e}^{(1)}) + x_2 \vec{f}(\hat{e}^{(2)}) + x_3 \vec{f}(\hat{e}^{(3)})$$

$$A = \begin{pmatrix} \uparrow & \uparrow & \uparrow \\ \vec{f}(\hat{e}^{(1)}) & \vec{f}(\hat{e}^{(2)}) & \vec{f}(\hat{e}^{(3)}) \\ \downarrow & \downarrow & \downarrow \end{pmatrix}$$

$$A\vec{x} = \begin{pmatrix} \uparrow & \uparrow & \uparrow \\ \vec{f}(\hat{e}^{(1)}) & \vec{f}(\hat{e}^{(2)}) & \vec{f}(\hat{e}^{(3)}) \\ \downarrow & \downarrow & \downarrow \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$= x_1 \vec{f}(\hat{e}^{(1)}) + x_2 \vec{f}(\hat{e}^{(2)}) + x_3 \vec{f}(\hat{e}^{(3)}) = \vec{f}(\vec{x})$$

Matrix Multiplication

- Matrix A represents a linear transformation $\vec{f}$
  - With respect to the standard basis
  - If we use a different basis, the matrix changes!

- Matrix multiplication $A\vec{x}$ **evaluates** $\vec{f}(\vec{x})$

What are they, *really*?

- Matrices are sometimes just tables of numbers.
- But they often have a deeper meaning.

Main Idea

A square $(n \times n)$ matrix can be interpreted as a compact representation of a linear transformation $\vec{f} : \mathbb{R}^n \to \mathbb{R}^n$.

What's more, if A represents $\vec{f}$, then $A\vec{x} = \vec{f}(\vec{x})$; that is, multiplying by A is the same as evaluating $\vec{f}$.

Example

$$\vec{x} = 3\hat{e}^{(1)} - 4\hat{e}^{(2)} = \begin{pmatrix} 3 \\ -4 \end{pmatrix}$$

$$\vec{f}(\hat{e}^{(1)}) = -\hat{e}^{(1)} + 3\hat{e}^{(2)} \Rightarrow \begin{pmatrix} -1 \\ 3 \end{pmatrix}$$

$$\vec{f}(\hat{e}^{(2)}) = 2\hat{e}^{(1)} \Rightarrow \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

$$\vec{f}(\vec{x}) = 3f(\vec{e}^{(1)}) - 4f(\vec{e}^{(2)})$$

$$A = \begin{pmatrix} -1 & 2 \\ 3 & 0 \end{pmatrix}$$

$$A\vec{x} = \begin{pmatrix} -1 & 2 \\ 3 & 0 \end{pmatrix}\begin{pmatrix} 3 \\ -4 \end{pmatrix} = \begin{pmatrix} -11 \\ 9 \end{pmatrix}$$

Note

- All of this works because we assumed $\vec{f}$ is **linear**.
- If it isn't, evaluating $\vec{f}$ isn't so simple.

Note

- All of this works because we assumed $\vec{f}$ is **linear**.
- If it isn't, evaluating $\vec{f}$ isn't so simple.
- Linear algebra = simple!

Matrices in Other Bases

- The matrix of a linear transformation wrt the **standard basis**:

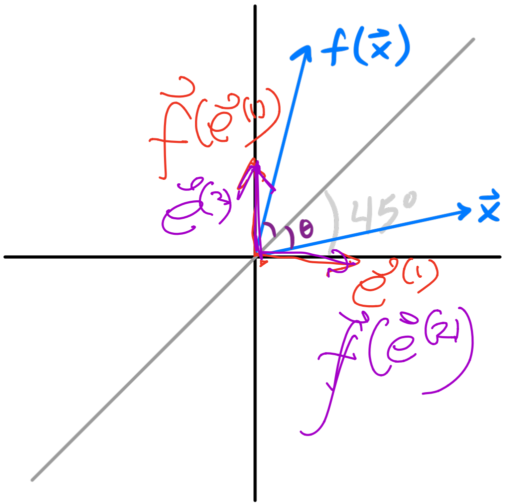
$$\begin{pmatrix} \uparrow & \uparrow & \uparrow & \\ \vec{f}(\hat{e}^{(1)}) & \vec{f}(\hat{e}^{(2)}) & \cdots & \vec{f}(\hat{e}^{(d)}) \\ \downarrow & \downarrow & \downarrow & \end{pmatrix}$$

- With respect to basis $\mathcal{U}$:

$$\begin{pmatrix} \uparrow & \uparrow & \uparrow & \\ [\vec{f}(\hat{u}^{(1)})]_{\mathcal{U}} & [\vec{f}(\hat{u}^{(2)})]_{\mathcal{U}} & \cdots & [\vec{f}(\hat{u}^{(d)})]_{\mathcal{U}} \\ \downarrow & \downarrow & \downarrow & \end{pmatrix}$$

Matrices in Other Bases

- Consider the transformation $\vec{f}$ which "mirrors" a vector over the line of 45°.

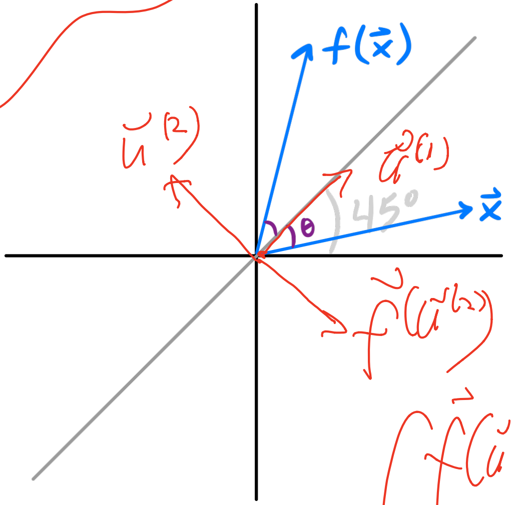


$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

- What is its matrix in the standard basis?

$$\left(\vec{f}(\vec{e}^{(1)}), \vec{f}(\vec{e}^{(2)}) \right)$$

Matrices in Other Bases



- Let $\hat{u}^{(1)} = \frac{1}{\sqrt{2}}(1,1)^T$
- Let $\hat{u}^{(2)} = \frac{1}{\sqrt{2}}(-1,1)^T$
- What is $[\vec{f}(\hat{u}^{(1)})]_{\mathcal{U}}$?
- $[\vec{f}(\hat{u}^{(2)})]_{\mathcal{U}}$?
- What is the matrix?

DSC 140B

Representation Learning

Lecture 05 | Part 2

The Spectral Theorem

Eigenvectors

- Let A be an $n \times n$ matrix. An **eigenvector** of A with **eigenvalue** λ is a nonzero vector $\vec{v}$ such that $A\vec{v} = \lambda\vec{v}$.

Eigenvectors (of Linear Transformations)

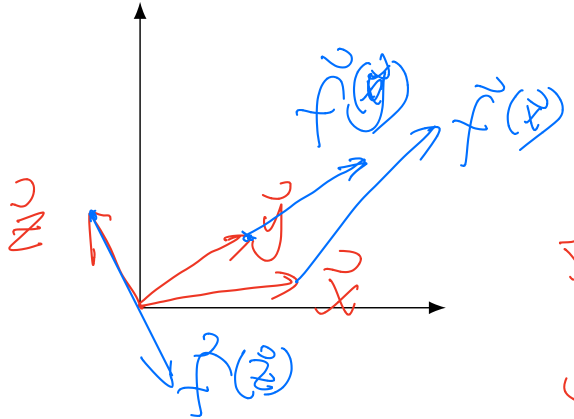
- Let $\vec{f}$ be a linear transformation. An **eigenvector** of $\vec{f}$ with **eigenvalue** λ is a nonzero vector $\vec{v}$ such that $f(\vec{v}) = \lambda\vec{v}$.

Importance

- We will see why eigenvectors are important in the next part.
- For now: what are they?

Geometric Interpretation

- When $\vec{f}$ is applied to one of its eigenvectors, $\vec{f}$ simply scales it.
  - Possibly by a negative amount.



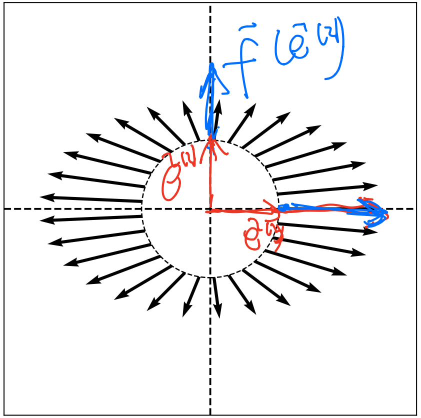
Exercise

Draw as many (linearly independent) eigenvectors as you can:

$A \cdot \vec{e}^{(1)} = 5\,\vec{e}^{(1)}$

$A \cdot \vec{e}^{(2)} = 2\,\vec{e}^{(2)}$

$$A = \begin{pmatrix} 5 & 0 \\ 0 & 2 \end{pmatrix}$$



$A\vec{e}^{(1)} = \lambda \vec{e}^{(1)}$

$n \times n$

$\leq n$

Finding Eigenvectors

- We typically compute the eigenvectors of a matrix with a computer.
- But it can help our understanding to find them "graphically".

Procedure

Given a matrix A (or transformation $\vec{f}$), to find an eigenvector "graphically".

1. Think about (or draw) the output of $\vec{f}$ for a handful of unit vector inputs.
   - Linear transformations are continuous so you can "interpolate".

2. Find place(s) where the input vector and the output vector are parallel.

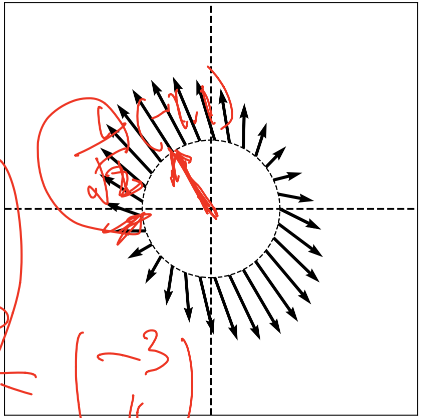
$\vec{f}(\vec{x}) = \lambda \vec{x}$

$\vec{f}(\alpha \cdot \vec{x}) = \lambda \alpha \vec{x}$

Exercise

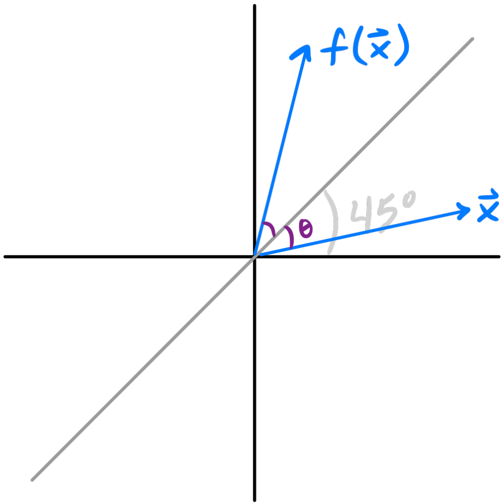
Draw as many (linearly independent) eigenvectors as you can:

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 3 \end{pmatrix}$$



Exercise

Consider the linear transformation which mirrors its input over the line of 45°. Give two orthogonal eigenvectors of the transformation.



Alternate Procedure: Guess and Check

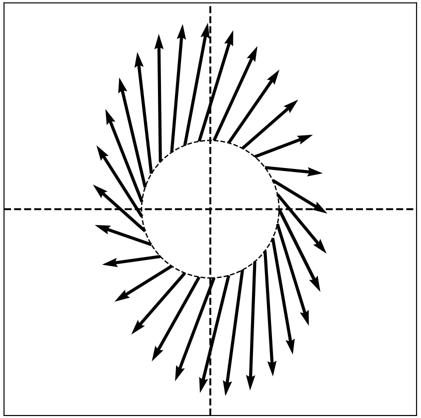
1. Guess a vector $\vec{x}$.

2. Check that $\vec{f}(\vec{x}) = \lambda\vec{x}$.

Exercise

Draw as many (linearly independent) eigenvectors as you can:

$$A = \begin{pmatrix} 5 & 5 \\ -10 & 12 \end{pmatrix}$$



Caution!

- Not all matrices have even one eigenvector![3]
- When does a matrix have multiple (linearly independent) eigenvectors?

[3] That is, with a *real-valued* eigenvalue.

Symmetric Matrices

- Recall: a matrix A is **symmetric** if $A^T = A$.

The Spectral Theorem[4]

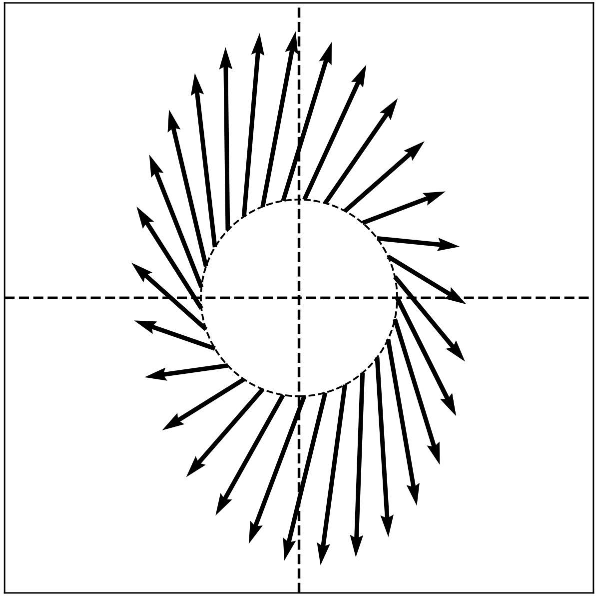
- **Theorem**: Let A be an $n \times n$ *symmetric* matrix. Then there exist n eigenvectors of A which are all mutually orthogonal.

[4] for symmetric matrices

What?

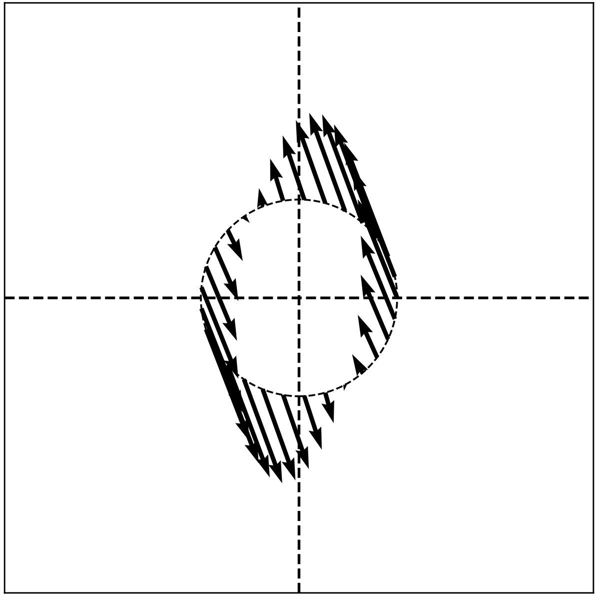
- What does the spectral theorem mean?
- What is an eigenvector, really?
- Why are they useful?

Example Linear Transformation



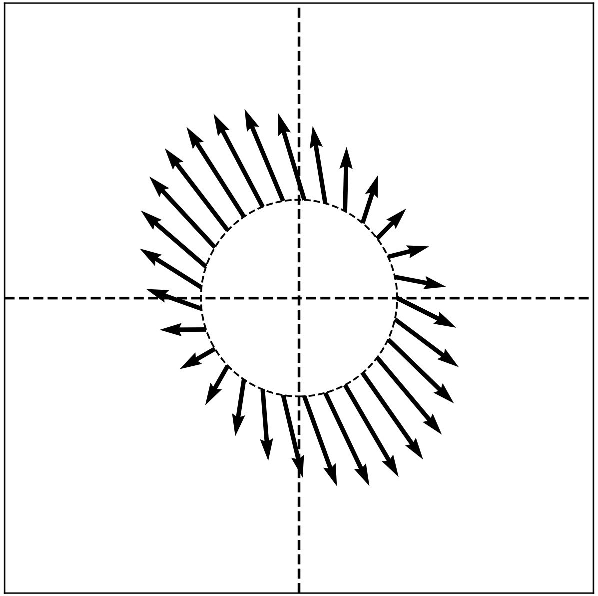
$$A = \begin{pmatrix} 5 & 5 \\ -10 & 12 \end{pmatrix}$$

Example Linear Transformation



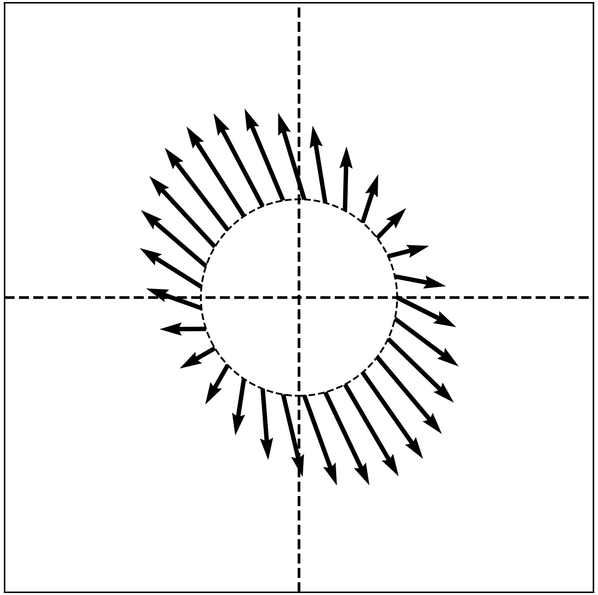
$$A = \begin{pmatrix} -2 & -1 \\ -5 & 3 \end{pmatrix}$$

Example Symmetric Linear Transformation



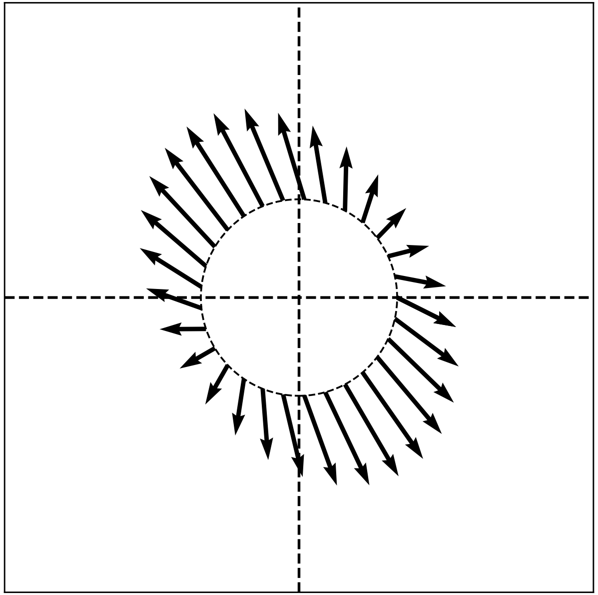
$$A = \begin{pmatrix} 2 & -1 \\ -1 & 3 \end{pmatrix}$$

Observation #1



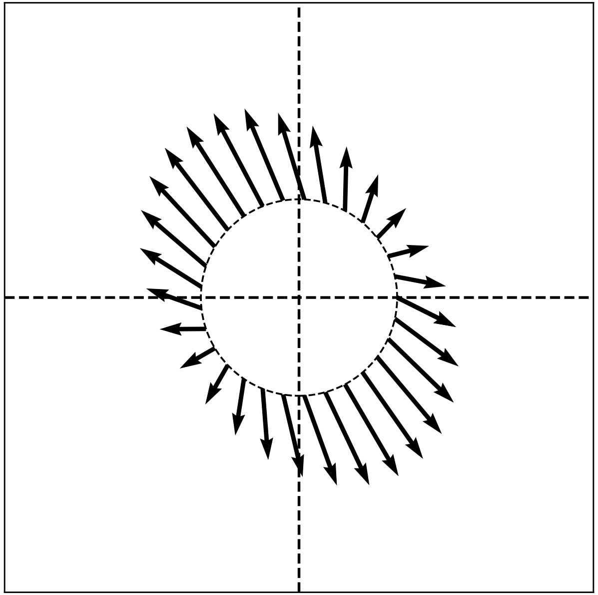
- Symmetric linear transformations have **axes of symmetry**.

Observation #2



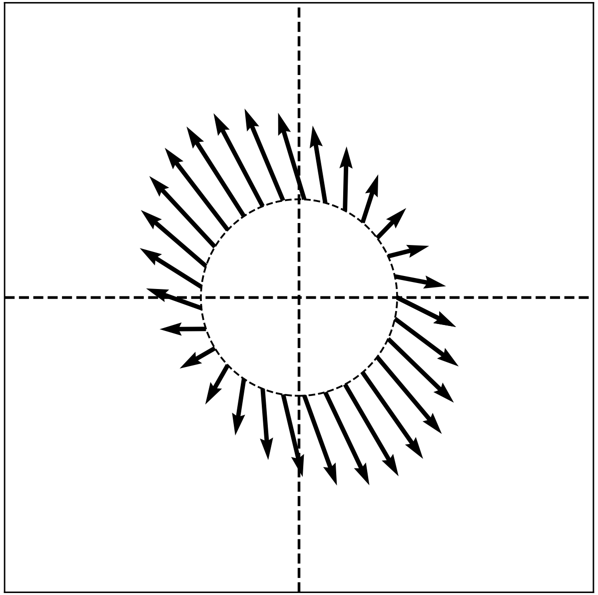
- The axes of symmetry are **orthogonal** to one another.

Observation #3



- The action of $\vec{f}$ along an axis of symmetry is simply to scale its input.

Observation #4



- The size of this scaling can be different for each axis.

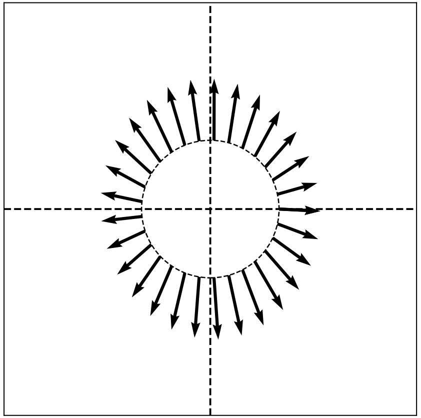
Main Idea

The **eigenvectors** of a symmetric linear transformation (matrix) are its axes of symmetry. The **eigenvalues** describe how much each axis of symmetry is scaled.

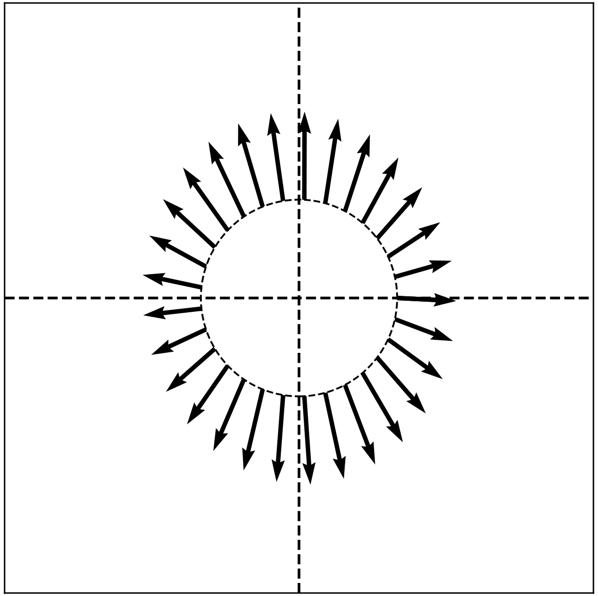
Diagonal Matrices

- If A is diagonal, its eigenvectors are simply the standard basis vectors.

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 5 \end{pmatrix}$$

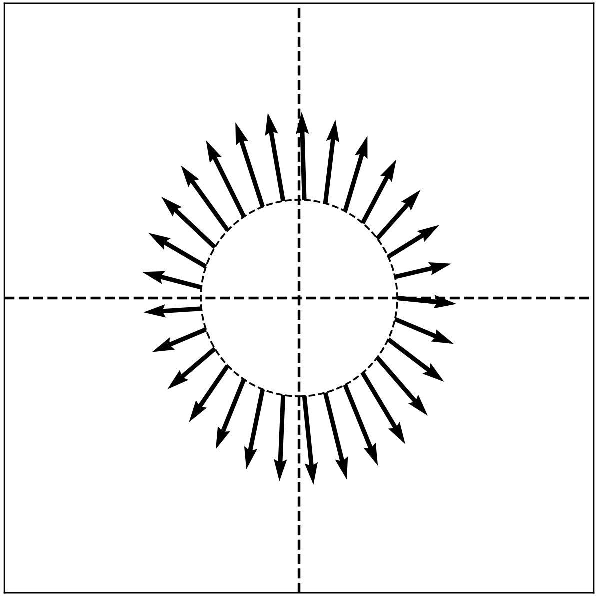


Off-diagonal elements



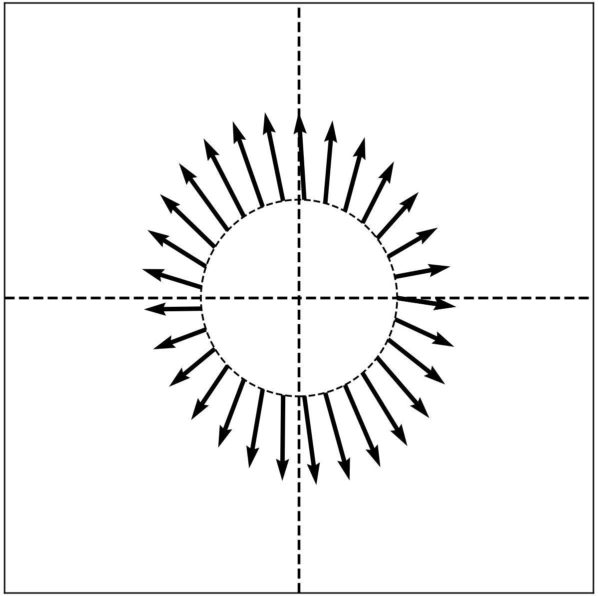
$$A = \begin{pmatrix} 2 & -0.1 \\ -0.1 & 5 \end{pmatrix}$$

Off-diagonal elements



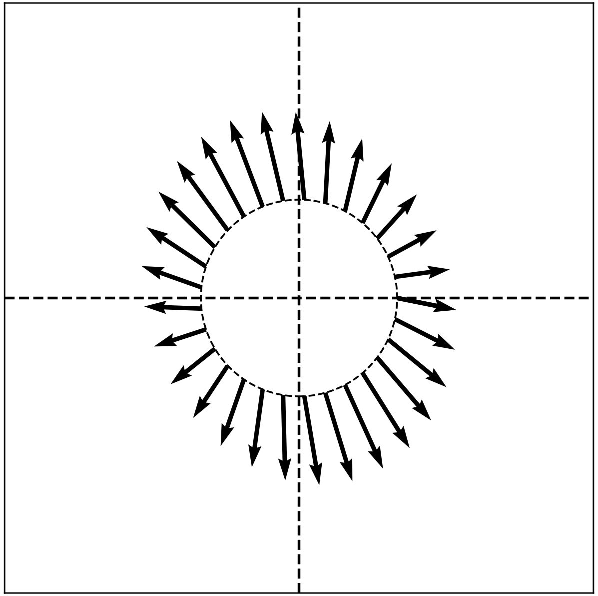
$$A = \begin{pmatrix} 2 & -0.2 \\ -0.2 & 5 \end{pmatrix}$$

Off-diagonal elements



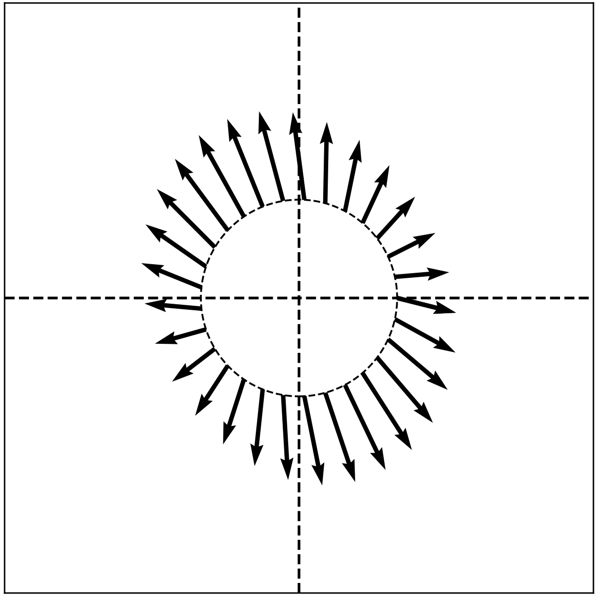
$$A = \begin{pmatrix} 2 & -0.3 \\ -0.3 & 5 \end{pmatrix}$$

Off-diagonal elements



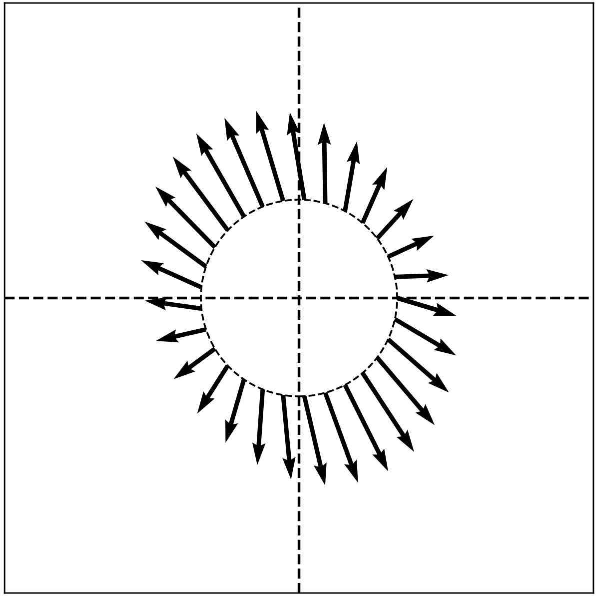
$$A = \begin{pmatrix} 2 & -0.4 \\ -0.4 & 5 \end{pmatrix}$$

Off-diagonal elements



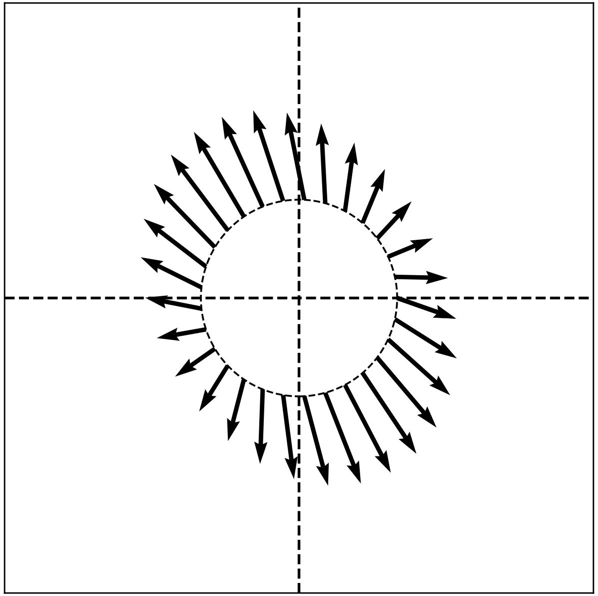
$$A = \begin{pmatrix} 2 & -0.5 \\ -0.5 & 5 \end{pmatrix}$$

Off-diagonal elements



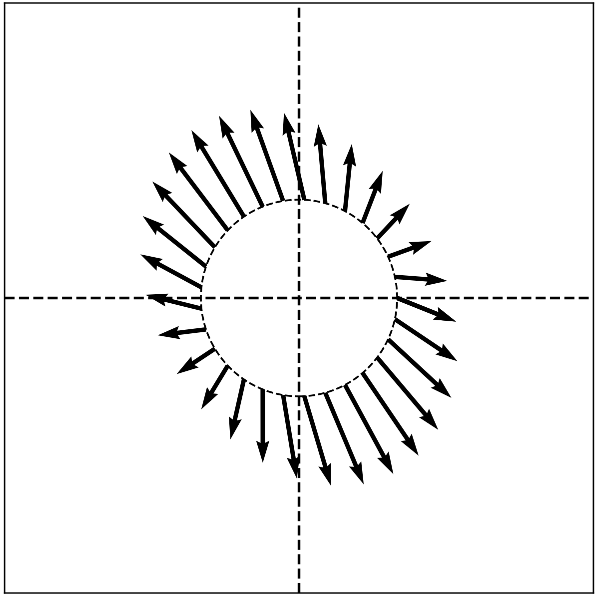
$$A = \begin{pmatrix} 2 & -0.6 \\ -0.6 & 5 \end{pmatrix}$$

Off-diagonal elements



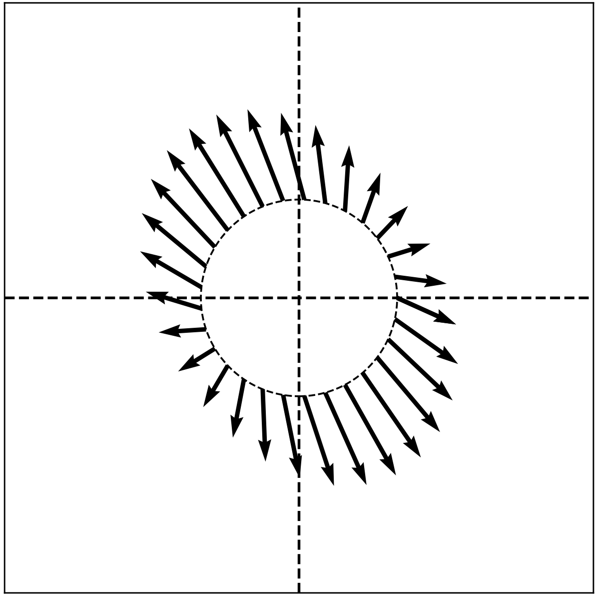
$$A = \begin{pmatrix} 2 & -0.7 \\ -0.7 & 5 \end{pmatrix}$$

Off-diagonal elements



$$A = \begin{pmatrix} 2 & -0.8 \\ -0.8 & 5 \end{pmatrix}$$

Off-diagonal elements



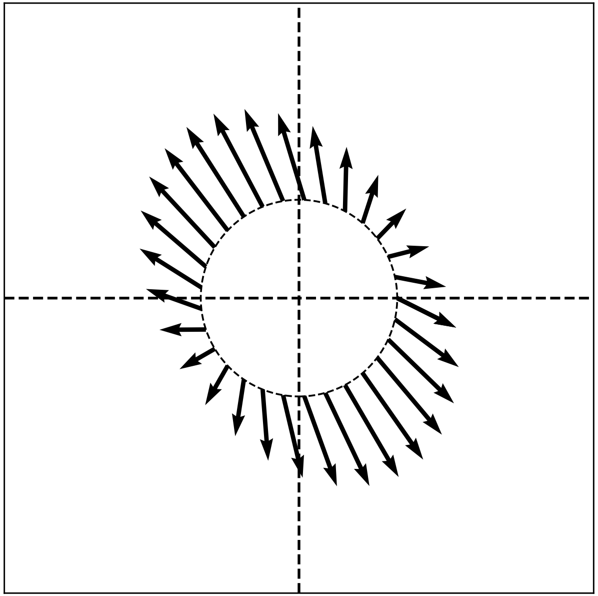
$$A = \begin{pmatrix} 2 & -0.9 \\ -0.9 & 5 \end{pmatrix}$$

Non-Diagonal Symmetric Matrices

- When a symmetric matrix is not diagonal, its eigenvectors are not the standard basis vectors.
- But they can be used to form an orthonormal basis!

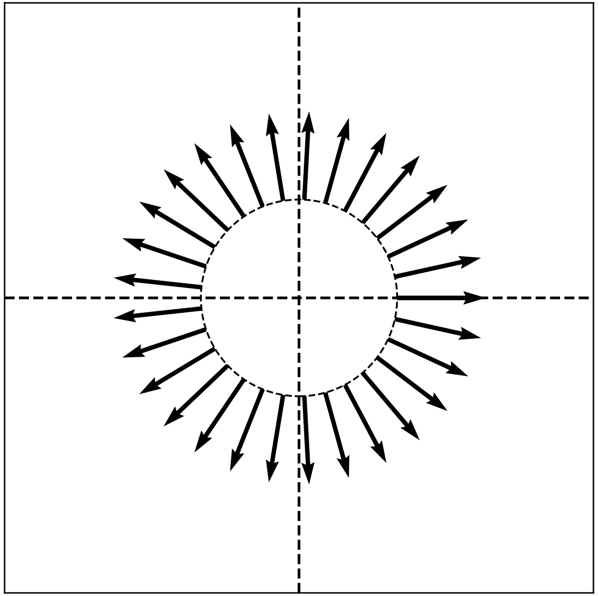
The Spectral Theorem[5]

- **Theorem**: Let A be an $n \times n$ *symmetric* matrix. Then there exist n eigenvectors of A which are all mutually orthogonal.



[5]for symmetric matrices

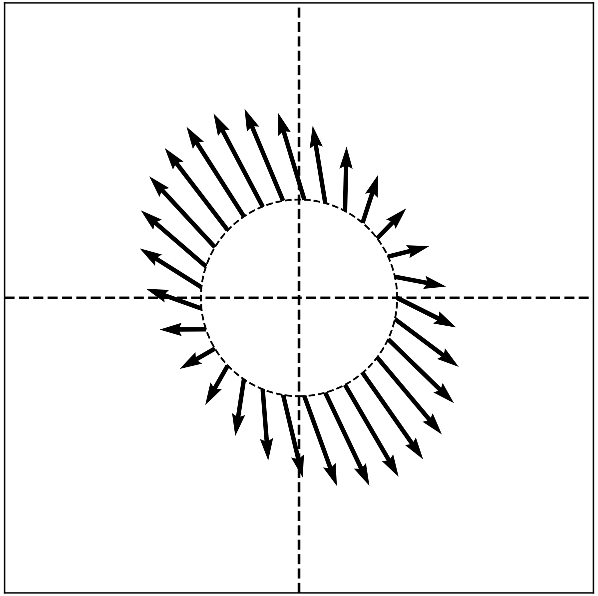
What about total symmetry?



- Every vector is an eigenvector.

$$A = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$$

Computing Eigenvectors



```
»> A = np.array([[2, -1], [-1, 3]])
»> np.linalg.eigh(A)
(array([1.38196601, 3.61803399]),
 array([[-0.85065081, -0.52573111],
        [-0.52573111,  0.85065081]]))
```