DSC 140B

Representation Learning

Lecture 04 | Part 1

Coordinate Vectors

Change of Basis

- Let $\mathcal{U} = \{\hat{u}^{(1)}, \ldots, \hat{u}^{(d)}\}$ be an orthonormal basis.
- The coordinates of $\vec{x}$ w.r.t. $\mathcal{U}$ are:

$$[\vec{x}]_{\mathcal{U}} = \begin{pmatrix} \vec{x} \cdot \hat{u}^{(1)} \\ \vec{x} \cdot \hat{u}^{(2)} \\ \vdots \\ \vec{x} \cdot \hat{u}^{(d)} \end{pmatrix}$$

$$[\vec{x}]_{\mathcal{U}} = \begin{pmatrix} 3/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix} \Leftarrow \begin{pmatrix} \vec{x} \cdot \hat{u}^{(1)} \\ \vec{x} \cdot \hat{u}^{(2)} \end{pmatrix}$$

Exercise

Suppose $\vec{x} = (2, 1)^T$ and let $\hat{u}^{(1)} = \frac{1}{\sqrt{2}}(1, 1)^T$ and $\hat{u}^{(2)} = \frac{1}{\sqrt{2}}(-1, 1)^T$. What is $[\vec{x}]_{\mathcal{U}}$?

$$\begin{pmatrix} -\frac{1}{13} \\ \frac{10}{13} \end{pmatrix}$$

$$\hat{u}^{(1)} = 3\vec{e}^{(1)} - 2\vec{e}^{(2)} = \begin{pmatrix} 3 \\ -2 \end{pmatrix} \qquad \hat{u}^{(2)} = \begin{pmatrix} -1 \\ 5 \end{pmatrix}$$

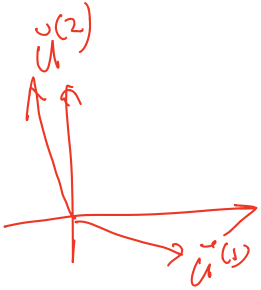
Exercise

Let $\vec{x} = (-1, 4)^T$ and suppose:

$$\hat{u}^{(1)} \cdot \hat{e}^{(1)} = 3 \qquad \hat{u}^{(2)} \cdot \hat{e}^{(1)} = -1$$

$$\hat{u}^{(1)} \cdot \hat{e}^{(2)} = -2 \qquad \hat{u}^{(2)} \cdot \hat{e}^{(2)} = 5$$

What is $[\vec{x}]_{\mathcal{U}}$?



$\vec{u}^{(1)}, \vec{u}^{(2)}$

not orthonormal

$$[\vec{x}]_{\mathcal{U}} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

$$\vec{x} = \alpha \hat{u}^{(1)} + \beta \hat{u}^{(2)}$$

$$\begin{pmatrix} -1 \\ 4 \end{pmatrix} = \alpha \begin{pmatrix} 3 \\ -2 \end{pmatrix} + \beta \begin{pmatrix} -1 \\ 5 \end{pmatrix}$$

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Lecture 04 | Part 2

Functions of a Vector

Functions of a Vector

- In ML, we often work with functions of a vector: $f : \mathbb{R}^d \to \mathbb{R}^{d'}$.
- Example: a prediction function, $H(\vec{x})$.
- Functions of a vector can return:
  - a number: $f : \mathbb{R}^d \to \mathbb{R}^1$
  - a vector $\vec{f} : \mathbb{R}^d \to \mathbb{R}^{d'}$
  - something else?

Transformations

- A **transformation** $\vec{f}$ is a function that takes in a vector, and returns a vector *of the same dimensionality*.
- That is, $\vec{f} : \mathbb{R}^d \to \mathbb{R}^d$.

Visualizing Transformations

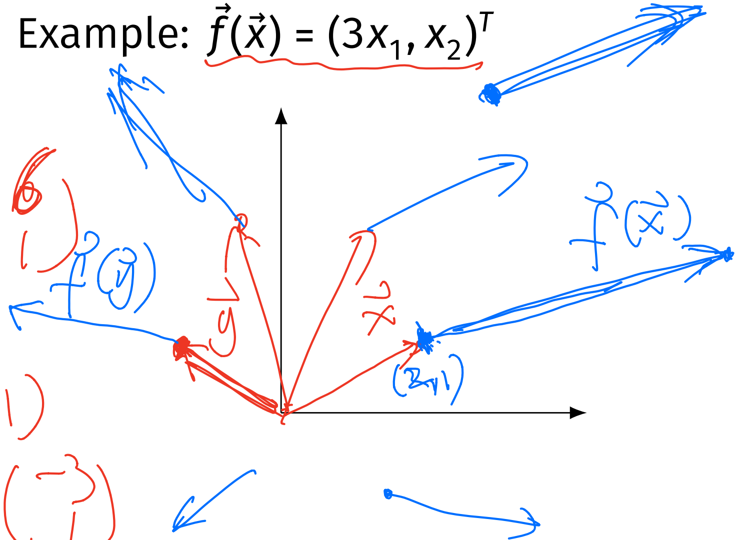
- A transformation is a **vector field**.
  - Assigns a vector to each point in space.
  - Example: $\vec{f}(\vec{x}) = (3x_1, x_2)^T$

$\vec{x} = (2, 1)$

$\vec{f}(\vec{x}) = \begin{pmatrix} 6 \\ 1 \end{pmatrix}$

$\vec{y} = (-1, 1)$

$\vec{f}(\vec{y}) = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$

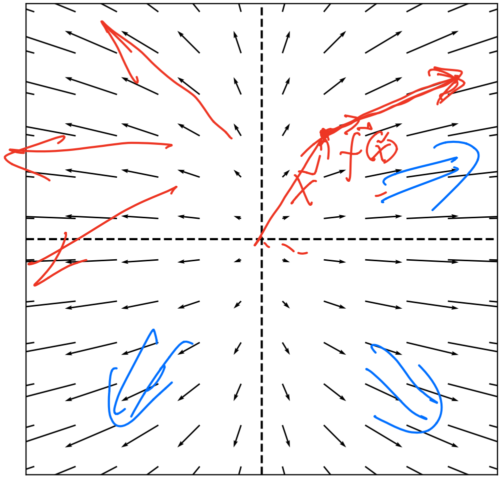


Example

- $\vec{f}(\vec{x}) = (3x_1, x_2)^T$

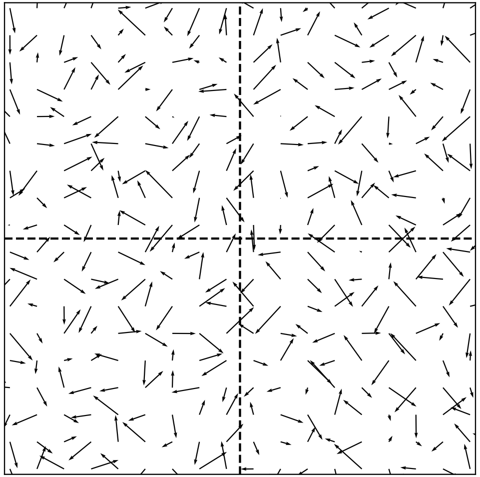
$\vec{x} = (1,2)$

$\vec{f}(\vec{x}) = (3,2)$



Arbitrary Transformations

- Arbitrary transformations can be quite complex.



Arbitrary Transformations

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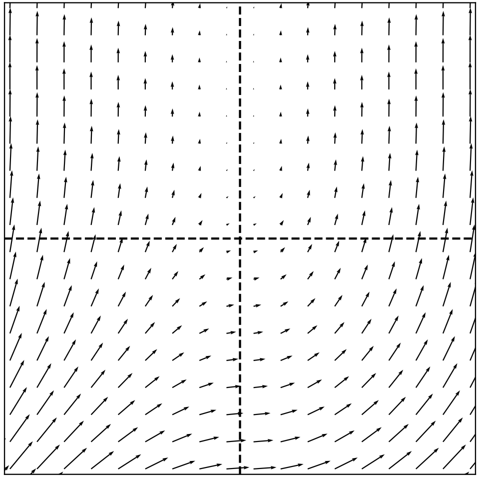


Linear Transformations

- Luckily, we often[1] work with simpler, **linear transformations**.

- A transformation f is linear if:

$$\vec{f}(\alpha\vec{x} + \beta\vec{y}) = \alpha\vec{f}(\vec{x}) + \beta\vec{f}(\vec{y})$$

[1]Sometimes, just to make the math tractable!

Checking Linearity

- To check if a transformation is linear, use the definition.
- **Example:** $\vec{f}(\vec{x}) = (x_2, -x_1)^T$

$\vec{x} = (x_1, x_2)^T$

$\vec{y} = (y_1, y_2)^T$

$\vec{f}(\alpha\vec{x} + \beta\vec{y}) \overset{?}{=} \alpha\vec{f}(\vec{x}) + \beta\vec{f}(\vec{y})$

$\vec{f}(\alpha\vec{x} + \beta\vec{y}) = \vec{f}\left(\alpha\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \beta\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\right)$

$= \vec{f}\left(\begin{pmatrix} \alpha x_1 + \beta y_1 \\ \alpha x_2 + \beta y_2 \end{pmatrix}\right)$

$= \begin{pmatrix} \alpha x_2 + \beta y_2 \\ -\alpha x_1 - \beta y_1 \end{pmatrix}$

$= \alpha \underbrace{\begin{pmatrix} x_2 \\ -x_1 \end{pmatrix}}_{\vec{f}(\vec{x})} + \beta \underbrace{\begin{pmatrix} y_2 \\ -y_1 \end{pmatrix}}_{\vec{f}(\vec{y})}$

Exercise

Let $\vec{f}(\vec{x}) = (x_1 + 3, x_2)$. Is $\vec{f}$ a linear transformation?

$$\vec{f}\left(3\underbrace{\begin{pmatrix}1\\0\end{pmatrix}}_{\vec{x}} + 4\underbrace{\begin{pmatrix}1\\1\end{pmatrix}}_{\vec{y}}\right) \neq 3\vec{f}\begin{pmatrix}1\\0\end{pmatrix} + 4\vec{f}\begin{pmatrix}1\\1\end{pmatrix}$$

Implications of Linearity

- Suppose $\vec{f}$ is a linear transformation. Then:

$$\begin{aligned}\vec{f}(\vec{x}) &= \vec{f}(x_1\hat{e}^{(1)} + x_2\hat{e}^{(2)}) \\ &= x_1\vec{f}(\hat{e}^{(1)}) + x_2\vec{f}(\hat{e}^{(2)})\end{aligned}$$

- I.e., $\vec{f}$ is **totally determined** by what it does to the basis vectors.

The Complexity of Arbitrary Transformations

- Suppose f is an **arbitrary** transformation.
- I tell you $\vec{f}(\hat{e}^{(1)}) = (2, 1)^T$ and $\vec{f}(\hat{e}^{(2)}) = (-3, 0)^T$.
- I tell you $\vec{x} = (x_1, x_2)^T$.
- What is $\vec{f}(\vec{x})$?

X

The Simplicity of Linear Transformations

- Suppose f is a **linear** transformation.
- I tell you $\vec{f}(\hat{e}^{(1)}) = (2, 1)^T$ and $\vec{f}(\hat{e}^{(2)}) = (-3, 0)^T$.
- I tell you $\vec{x} = (x_1, x_2)^T$.
- What is $\vec{f}(\vec{x})$?

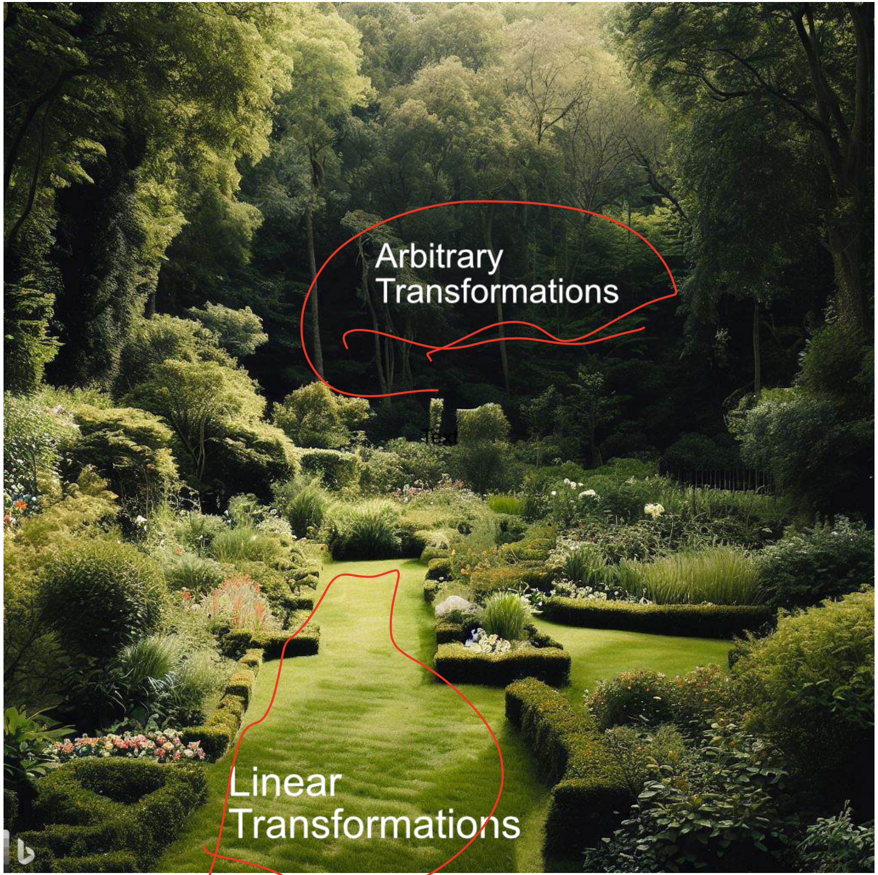
$$f(\vec{x}) = \vec{f}\left(\begin{pmatrix}3\\-4\end{pmatrix}\right) = 3\vec{f}(\hat{e}^{(1)}) - 4\vec{f}(\hat{e}^{(2)}) = \begin{pmatrix}18\\3\end{pmatrix}$$

Exercise

- Suppose f is a **linear** transformation.
- I tell you $\vec{f}(\hat{e}^{(1)}) = (2, 1)^T$ and $\vec{f}(\hat{e}^{(2)}) = (-3, 0)^T$.
- I tell you $\vec{x} = (3, -4)^T$.
- What is $\vec{f}(\vec{x})$?

Key Fact

- Linear functions are determined **entirely** by what they do on the basis vectors.
- I.e., to tell you what f does, I only need to tell you $\vec{f}(\hat{e}^{(1)})$ and $\vec{f}(\hat{e}^{(2)})$.
- This makes the math easy!



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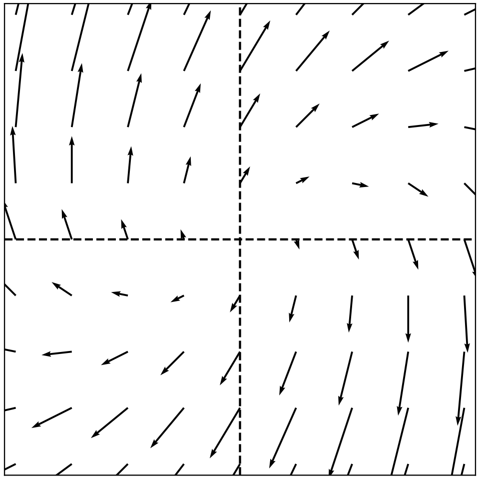
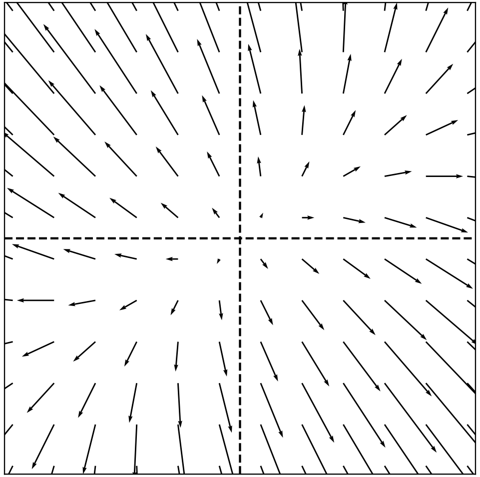
Example Linear Transformation

- $\vec{f}(\vec{x}) = (x_1 + 3x_2, -3x_1 + 5x_2)^T$



Another Example Linear Transformation

- $\vec{f}(\vec{x}) = (2x_1 - x_2, -x_1 + 3x_2)^T$

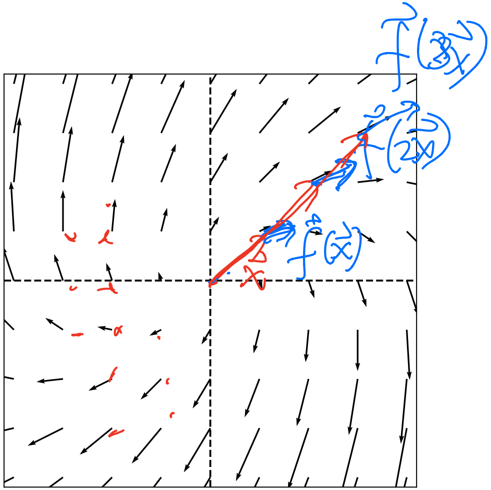


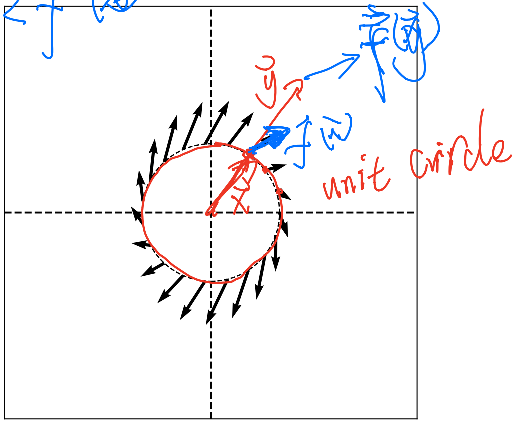
Note

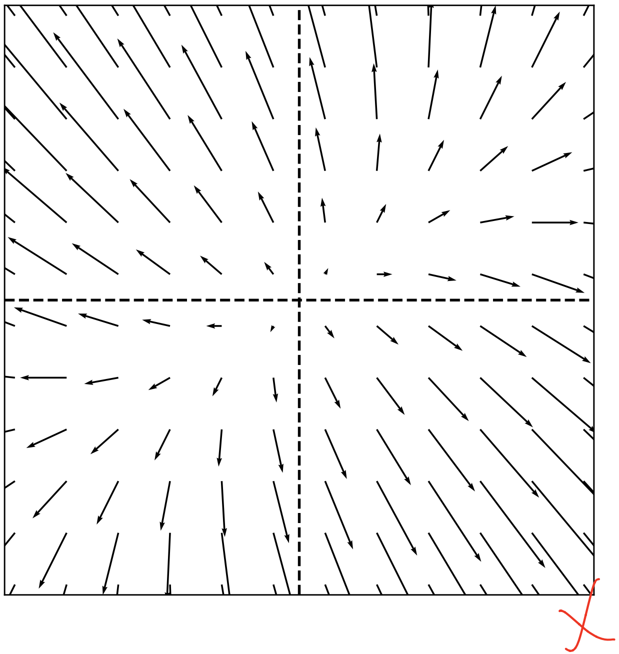
- Because of linearity, along any given direction $\vec{f}$ changes only in scale.

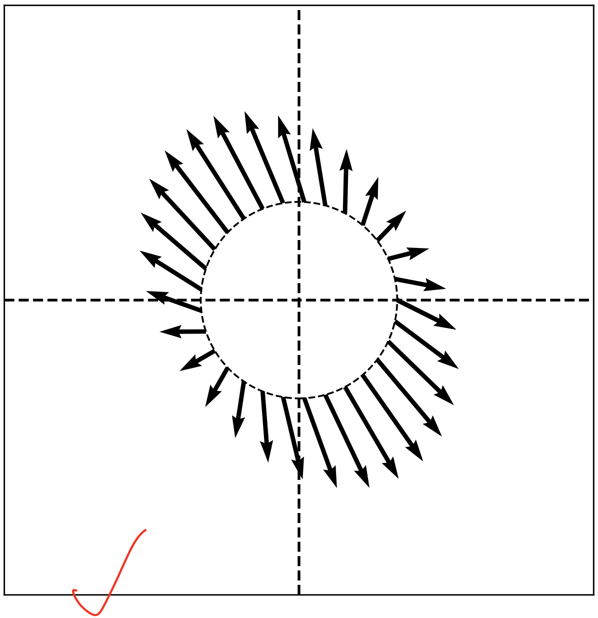
$$\vec{f}(\lambda\hat{x}) = \lambda\vec{f}(\hat{x})$$

$\vec{f}(2\vec{x}) = 2\vec{f}(\vec{x})$









Linear Transformations and Bases

- We have been writing transformations in coordinate form. For example:

$$\vec{f}(\vec{x}) = (x_1 + x_2, x_1 - x_2)^T$$

- To do so, we assumed the **standard basis**.
- If we use a different basis, the formula for $\vec{f}$ changes.

Example

- Suppose that in the standard basis, $\vec{f}(\vec{x}) = (x_1 + x_2, x_1 - x_2)^T$. linear transformation
- Let $\hat{u}^{(1)} = \frac{1}{\sqrt{2}}(1, 1)^T$ and $\hat{u}^{(2)} = \frac{1}{\sqrt{2}}(-1, 1)^T$. $\rightarrow$ orthonormal basis
- Write $[\vec{x}]_{\mathcal{U}} = (z_1, z_2)^T$. $\vec{x} = z_1 \vec{u}^{(1)} + z_2 \vec{u}^{(2)}$
- What is $[\vec{f}(\vec{x})]_{\mathcal{U}}$ in terms of z_1 and z_2?

$$\vec{f}(\vec{x}) = \vec{f}(z_1 \vec{u}^{(1)} + z_2 \vec{u}^{(2)})$$
$$= z_1 \vec{f}(\vec{u}^{(1)}) + z_2 \vec{f}(\vec{u}^{(2)})$$

$$[\vec{f}(\vec{x})]_u = \begin{pmatrix} \vec{f}(\vec{x}) \cdot \vec{u}^{(1)} \\ \vec{f}(\vec{x}) \cdot \vec{u}^{(2)} \end{pmatrix} = \begin{pmatrix} z_1 - z_2 \\ -z_1 - z_2 \end{pmatrix}$$

Example

- Suppose that in the standard basis, $\vec{f}(\vec{x}) = (x_1 + x_2, x_1 - x_2)^T$.
- Let $\hat{u}^{(1)} = \frac{1}{\sqrt{2}}(1, 1)^T$ and $\hat{u}^{(2)} = \frac{1}{\sqrt{2}}(-1, 1)^T$.
- Write $[\vec{x}]_{\mathcal{U}} = (z_1, z_2)^T$.
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Example

- Suppose that in the standard basis, $\vec{f}(\vec{x}) = (x_1 + x_2, x_1 - x_2)^T$.
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- What is $[\vec{f}(\vec{x})]_{\mathcal{U}}$ in terms of z_1 and z_2?